EXOTIC LEFT ORDERINGS OF THE FREE GROUPS FROM THE DEHORNOY ORDERING

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ABSTRACT. We show that the restriction of the Dehornoy ordering to an appropriate free subgroup of the three-strand braid group defines a left ordering of the free group on k generators, $k > 1$, that has no convex subgroups.

A group G is said to be left-orderable if there exists a strict total ordering of its elements such that $g < h$ implies $fg < fh$ for all f, g, h in G. To each left ordering $<$ of a group G, we can associate the set $P = \{g \in G | g > 1\}$, which is called the positive cone associated to the left ordering $<$. The positive cone P satisfies $P \cdot P \subset P$, and $P \sqcup P^{-1} \sqcup \{1\} = G$. Conversely, any subset P satisfying these two properties defines a strict total ordering of the elements of G, via $g < h$ if and only if $g^{-1}h \in P$. Any ordering defined in this way is easily seen to be invariant under left multiplication.

We may strengthen our conditions on a left ordering $<$ of G by requiring that for all $g, h > 1$ in G, there must exist a positive integer n such that $g < hg^n$. In this case, the ordering is called Conradian (after the work of Conrad in [2]). It has since been observed that, equivalently, we may ask that this condition hold for $n = 2$ [9].

Finally, the strongest condition we may require of an ordering $<$ of G is that the ordering be invariant under multiplication from both sides, that is, $g < h$ implies $fg < fh$ and $gf < hf$ for all f, g, h in G. Equivalently, we may require that the positive cone associated to the ordering $<$ of G be preserved by conjugation. If either of these equivalent conditions is satisfied by the ordering $<$ of G, then the ordering is said to be a bi-ordering.

An important structure associated to a given left ordering $<$ of G is the set of convex subgroups of G. A subgroup $H \subset G$ is said to be convex in G (with respect to the ordering $<$) if whenever f, h are in H and g is in G, the implication $f < g < h \Rightarrow g \in H$ holds.

Owing to work of Conrad and Hölder, the convex subgroups of bi-orderings and Conradian orderings are very well understood [2]. This leaves us with understanding the set of convex subgroups for the case of left orderings that are neither bi-orderings, nor Conradian orderings. This problem seems to be quite difficult, as constructing Conradian orderings and bi-orderings of a group G is in general somewhat easier than constructing left orderings of a group that are not Conrad orderings.

Date: October 25, 2009.

The two primary methods for constructing non-Conradian orderings of a group G are given by the following proposition and theorem.

Proposition 0.1. *Let G be a group, K a subgroup of G left ordered by the ordering $\prec$, and G/K the set of left cosets of K in G. Suppose that G/K is ordered by the ordering $\prec'$, satisfying $gK \prec' hK$ implies $fgK \prec' fhK$ for all f, g, h in G. Then a left ordering $<$ can be defined on G, according to the rule: For every g in G, $1 < g$ if $g \in K$ and $1 \prec g$, or if $g \notin K$ and $K \prec' gK$.*

Theorem 0.2. *(Conrad,* [2]*) A group G is left-orderable if and only if G acts effectively by order preserving automorphisms on a linearly ordered set.*

In both of these cases, at least some of the convex subgroups of the constructed ordering are obvious. In Proposition 0.1, the subgroup K is a convex subgroup in the left ordering $<$ of G. In Theorem 0.2, the stabilizers under the G-action of points in the given linearly ordered set correspond to convex subgroups (see [2] or [9] for details of the construction). In light of the fact that both of these known methods for producing left orderings of a group result in an ordering that (often) contain convex subgroups, it is quite surprising to find that some non-Conradian left orderings may contain no proper, nontrivial convex subgroups whatsoever. In this paper, we will left order the free groups of finite rank in a way so that the free group contains no proper, nontrivial convex subgroups with respect to our constructed ordering. The construction relies heavily on the Dehornoy ordering of the braid group B_3.

The existence and a construction of such orderings of the free groups seems to have appeared only in [7]. The construction there, unlike our present setting, deals with creating a very unusual effective action on the rationals. Our present approach is in simpler algebraic terms.

It is also worth noting that admitting a Conradian or bi-ordering that has no proper, nontrivial convex subgroups is a very restrictive condition on the group G, as the following theorem shows.

Theorem 0.3. [2] *Suppose that G admits a Conradian or bi-ordering which has no proper, nontrivial convex subgroups. Then G is a subgroup of $(\mathbb{R}, +)$.*

In the case that G admits a non-Conradian left ordering having no proper, nontrivial convex subgroups, it is not likely that the structure of G must be so restricted. While we will see that free groups admit such left orderings, there are also non-free, non-Abelian groups that admit such left orderings as well ([1] Example 7.2.3). It has also recently been shown that the braid groups themselves admit many left-orderings with no convex subgroups, see [10].

1. A left ordering of F_2 having no convex subgroups.

As a warm up for the general case, which will be slightly more involved, we deal first with the free group on two generators.

We begin by defining the Dehornoy left ordering of the braid groups (also known as the 'standard' ordering), whose positive cone we shall denote P_D [4], [3]. Recall that for each integer $n \geq 2$, the Artin braid group B_n is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1.$$

Definition 1.1. *Let w be a word in the generators $\sigma_i, \cdots, \sigma_{n-1}$. Then w is said to be: i-positive if the generator σ_i occurs in W with only positive exponents, i-negative if σ_i occurs with only negative exponents, and i-neutral if σ_i does not occur in w.*

We then define the positive cone of the Dehornoy ordering as

Definition 1.2. *The positive cone $P_D \subset B_n$ of the Dehornoy ordering is the set*

$$P_D = \{\beta \in B_n : \ \beta \textit{ is } i\textit{-positive for some } i \leq n-1\}.$$

Let $\beta \in B_n$ be any braid. An extremely important property of this ordering is that the conjugate $\beta\sigma_k\beta^{-1}$ is always i-positive for some i, for every generator σ_k in B_n. This property is referred to as the subword property [4].

Recall that the commutator subgroup $[B_3, B_3]$ is isomorphic to the free group F_2 on two generators. The commutator subgroup is generated by the braids $\beta_1 = \sigma_2\sigma_1^{-1}$ and $\beta_2 = \sigma_1\sigma_2\sigma_1^{-2}$ [8]. Of course we can instead take as generators the 1-positive braids $\beta_1^{-1} = \sigma_1\sigma_2^{-1}$ and $\beta_2^{-1}\beta_1^{-1} = \sigma_1^2\sigma_2^{-2}$.

Define a positive cone $P \subset F_2$ by $P = [B_3, B_3] \cap P_D$, with associated ordering $<$ of F_2. Thus, the ordering $<$ of F_2 is the restriction of the Dehornoy ordering $<_D$ of B_3 to the (free) commutator subgroup $[B_3, B_3]$.

Theorem 1.3. *The ordering $<$ of F_2 has no proper, non-trivial convex subgroups.*

Proof. Let $C \subset F_2 = [B_3, B_3]$ be a nontrivial convex subgroup. Then we may choose $1 < \beta \in F_2$ that is 1-positive (no nontrivial 1-neutral braids lie in $[B_3, B_3]$, because they do not have zero total exponent).

There are now two cases to consider.

Case 1. Suppose that β commutes with σ_2. Then $\beta = \Delta_3^{2p}\sigma_2^q$ for some integers p, q ([5], here $\Delta = \sigma_1\sigma_2\sigma_1$). Since $\beta \in [B_3, B_3]$, we know that $q = -6p$, since β must have zero total exponent, and $p > 0$ because we have chosen β to be 1-positive. Then we have that $\Delta_3^2 < \Delta_3^{4p}\sigma_2^{-12p} = \beta^2$, so that β is cofinal in the Dehornoy ordering [4]. Therefore, there exist integers k, l so that in F_2 we have

$$1 < \sigma_1\sigma_2^{-1} < \beta^k, \text{ and } 1 < \sigma_1^2\sigma_2^{-2} < \beta^l,$$

and thus $\sigma_1\sigma_2^{-1}, \sigma_1^2\sigma_2^{-2} \in C$ by convexity. Therefore we must have $C = F_2$, as C contains both generators of F_2.

Case 2. Suppose that β and σ_2 do not commute. Let $k > 0$, and observe that $\beta\sigma_2^k\beta^{-1}$ is a 1-positive braid by the subword property, so

that the commutator $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is also 1-positive. Next, because β is 1-positive, the braid $\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is 1-negative, so that $\sigma_2^k\beta^{-1}\sigma_2^{-k} < 1$, and thus $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$. Thus, we have shown that $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$, so that $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ must lie in the subgroup C, by convexity.

Now both the braids β and $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ lie in the convex subgroup C, so the braid $\sigma_2^k\beta^{-1}\sigma_2^{-k}$ (and hence its inverse $\sigma_2^k\beta\sigma_2^{-k}$) must also lie in C, for any choice of positive integer k.

We now refine our choice of braid $\beta \in C$. Suppose that β is represented by the 1-positive braid word $\sigma_2^u\sigma_1 w$, where u is any integer, and w is a 1-positive, 1-neutral or empty word. Choose $k > 0$ so that $u' = k + u > 0$, and set $\beta' = \sigma_2^k\beta\sigma_2^{-k}$, so that β' is represented by the 1-positive braid word $\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$. Note that $\beta' \in C$, from our work above.

We will now show that C must contain both generators of F_2. Observe that the braid represented by the word $\sigma_2\sigma_1^{-1}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$ is 1-positive, as $\sigma_2(\sigma_1^{-1}\sigma_2^{u'}\sigma_1)w\sigma_2^{-k} = \sigma_2(\sigma_2\sigma_1^{u'}\sigma_2^{-1})w\sigma_2^{-k}$, and $u' > 0$. Therefore we have

$$1 < \sigma_2\sigma_1^{-1}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k} \Rightarrow \sigma_1\sigma_2^{-1} < \sigma_2^{u'}\sigma_1 w\sigma_2^{-k} = \beta' \in C,$$

and since $1 < \sigma_1\sigma_2^{-1}$, this implies that $\sigma_1\sigma_2^{-1} \in C$ by convexity.

Considering the second generator $\sigma_1^2\sigma_2^{-2}$, observe that the braid represented by the word $\sigma_2^2\sigma_1^{-2}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$ is 1-positive, as we compute

$$\sigma_2^2\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{u'}\sigma_1)w\sigma_2^{-k} = \sigma_2^2\sigma_1^{-1}(\sigma_2\sigma_1^{u'}\sigma_2^{-1})w\sigma_2^{-k},$$

and

$$\sigma_2^2(\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^{u'-1}\sigma_2^{-1}w\sigma_2^{-k} = \sigma_2^2(\sigma_2\sigma_1\sigma_2^{-1})\sigma_1^{u'-1}\sigma_2^{-1}w\sigma_2^{-k},$$

where $u' > 0$. Therefore we have

$$1 < \sigma_2^2\sigma_1^{-2}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k} \Rightarrow \sigma_1^2\sigma_2^{-2} < \sigma_2^{u'}\sigma_1 w\sigma_2^{-k} = \beta' \in C,$$

and since $1 < \sigma_1^2\sigma_2^{-2}$, we conclude from convexity of C that $\sigma_1^2\sigma_2^{-2} \in C$.

Thus, C contains both generators of F_2, so that $C = F_2$.

$\square$

2. Left ordering the free groups of rank greater than two

We now extend our results to cover those free groups F_k with $k > 2$. Let $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$ denote the generators of F_2, and we let K_n denote the kernel of the map $F_2 \to \mathbb{Z}_{n-1}$ defined by $y \mapsto 0$, $x \mapsto 1$. Here $\mathbb{Z}_{n-1}$ is the cyclic group of order $n-1$. We will employ a proof very similar to that of Theorem 1.3, by considering $K_n \subset F_2 = [B_3, B_3]$, and showing that the restriction of the Dehornoy ordering to K_n has no convex subgroups. First we need to find a generating set for K_n.

Lemma 2.1. *The subgroup K_n is free of rank n, with basis*

$$y, x^{n-1}, xyx^{n-2}, x^2yx^{n-3}, \cdots, x^{n-2}yx.$$

Proof. From Lemma 7.56 of [11], we know that K_n is finitely generated. Moreover, we may compute a generating set of K_n as follows: Consider the generating set $g_1 = x, g_2 = x^{-1}, g_3 = y, g_4 = y^{-1}$ of F_2, and let $1, x, x^2, \cdots, x^{n-2}$ be representatives of the right cosets of $K_n \subset F_2$. For all i, j, there exists h_{ij} and some coset representative $x^{k(i,j)}$ such that we may write $x^i g_j = h_{ij} x^{k(i,j)}$. The elements h_{ij} form a generating set for K_n.

In our present setting, we find for $i < n-2$

$$x^i g_1 = x^i \cdot x = 1 \cdot x^{i+1},$$

so that $h(i,1) = 1$, and for $i = n-2$ we get $h(i,1) = x^{n-1}$. Similarly, we compute for $i \geq 1$ that

$$x^i g_2 = x^i \cdot x^{-1} = 1 \cdot x^{i-1},$$

so that $h(i,2) = 1$, and for $i = 0$ we compute $h(i,2) = x^{-(n-1)}$.

Next, for all i we compute

$$x^i y^{\pm 1} = x^i y^{\pm 1} x^{-i} \cdot x^i,$$

so that $h(i,3) = h(i,4)^{-1} = x^i y x^{-i}$. Eliminating inverses from this generating set yields the set

$$y, x^{n-1}, xyx^{-1}, x^2yx^{-2}, \cdots, x^{n-2}yx^{-(n-2)}.$$

From Proposition 3.9 of [6] we deduce that K_n is of rank n, and therefore the generating set above must provide a basis for K_n. Right multiplying those generators of the form $x^i y x^{-i}$ by the generator x^{n-1} yields the desired generating set. □

Also important in the proof of Theorem 1.3 was the action of conjugation by σ_2. In order to generalize our theorem, we must make the following analysis.

Let F_2 be the free group on two generators x and y, and define an automorphism $\phi : F_2 \to F_2$ according to the forumlas $\phi(x) = xy^{-1}x$, and $\phi(y) = xy^{-1}x^2$. Then the following holds.

Lemma 2.2. *Consider F_2 as the commutator subgroup $[B_3, B_3]$ with generators $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$. Then the automorphism ϕ of F_2 corresponds to conjugation of $[B_3, B_3]$ by the generator $\sigma_2 \in B_3$, so that $\phi(g) = \sigma_2^{-1} g \sigma_2$ for all $g \in F_2$.*

Proof. The proof is computational. First conjugating the generator x, we compute

$$\begin{aligned}
\phi(x) &= xy^{-1}x \\
&= \sigma_1\sigma_2^{-1}\sigma_2^2\sigma_1^{-2}\sigma_1\sigma_2^{-1} \\
&= (\sigma_1\sigma_2\sigma_1^{-1})\sigma_2^{-1} \\
&= (\sigma_2^{-1}\sigma_1\sigma_2)\sigma_2^{-1} \\
&= \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_2 \\
&= \sigma_2^{-1}x\sigma_2
\end{aligned}$$

and

$$\begin{aligned}
\phi(y) &= xy^{-1}x^2 \\
&= \sigma_1\sigma_2^{-1}\sigma_2^2\sigma_1^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\
&= (\sigma_1\sigma_2\sigma_1^{-1})\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\
&= (\sigma_2^{-1}\sigma_1\sigma_2)\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\
&= \sigma_2^{-1}\sigma_1^2\sigma_2^{-1} \\
&= \sigma_2^{-1}\sigma_1^2\sigma_2^{-2}\sigma_2 \\
&= \sigma_2^{-1}y\sigma_2.
\end{aligned}$$

□

This computation allows us to show that K_n is fixed by the conjugation action of σ_2^6 or σ_2^{-6} on the commutator subgroup $[B_3, B_3]$.

Lemma 2.3. *Let $\phi : F_2 \to F_2$ be the map arising from conjugation of $[B_3, B_3]$ by σ_2, namely $\phi(x) = xy^{-1}x$, and $\phi(y) = xy^{-1}x^2$. Then for all n, $\phi^6(K_n) = K_n$.*

Proof. Consider the abelianization $F_2 \xrightarrow{ab} \mathbb{Z} \oplus \mathbb{Z}$. We find that ϕ descends to a map $\phi_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, and that relative to the basis $\{ab(x), ab(y)\}$ the map ϕ_* is represented by the matrix

$$\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}.$$

The sixth power of this matrix is the identity. It follows that for any normal subgroup K such that F_2/K is abelian, we have $\phi^6(K) = K$. □

Lastly, note that any generator of K_n, when we substitute $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$, yields a product of braid group generators of the form $\sigma_1^{l_1}\sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_{m-1}}\sigma_1^{l_m}\sigma_2^{k_m}$, where $k_i < 0$ and $l_i > 0$ for all i. Therefore, we require the following lemma in order to compare the generators to different braids in K_n.

Lemma 2.4. *Any braid represented by a word of the form*
$$\sigma_2^{k_1}\sigma_1^{l_1}\cdots\sigma_2^{k_m}\sigma_1^{l_m}\sigma_2^{n}\sigma_1,$$
where $k_i > 0$, $l_i < 0$ for all i, and $n > 1$, is 1-positive.

Proof. We use induction on m, the length of the product. For $m = 0$, the claim is trivial. Assuming the claim holds for those products of length $m-1$, we use the identities $\sigma_1^k\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^k$ and $\sigma_1^{-1}\sigma_2^k\sigma_1 = \sigma_2\sigma_1^k\sigma_2^{-1}$, and compute that

$$\begin{aligned}
\sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m}\sigma_2^{n}\sigma_1 &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m+1}(\sigma_1^{-1}\sigma_2^{n}\sigma_1)\\
&= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m+1}(\sigma_2\sigma_1^{n}\sigma_2^{-1})\\
&= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}(\sigma_1^{l_m+1}\sigma_2\sigma_1)\sigma_1^{n-1}\sigma_2^{-1}\\
&= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}(\sigma_2\sigma_1\sigma_2^{l_m+1})\sigma_1^{n-1}\sigma_2^{-1}\\
&= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m+1}\sigma_1(\sigma_2^{l_m+1}\sigma_1^{n-1}\sigma_2^{-1}).
\end{aligned}$$

The bracketed expression $\sigma_2^{l_m+1}\sigma_1^{n-1}\sigma_2^{-1}$ is 1-positive as $n > 1$, and the remaining terms in the product above are representative of a 1-positive braid, by assumption. By induction, the claim is proven. □

Theorem 2.5. *Let $n > 2$. Then the restriction of the Dehornoy ordering to the subgroup $K_n \subset F_2 = [B_3, B_3]$ has no proper, nontrivial convex subgroups.*

Proof. We proceed similarly to Theorem 1.3. Suppose that $C \subset K_n$ is a nontrivial, convex subgroup, and let $\beta \in C$ be a 1-positive braid. Denote the generators of K_n by $g_1, g_2, \cdots, g_n$, from Lemma 2.1 we know that $g_i > 1$ for all i. There are two cases to consider.

Case 1. The braid β commutes with σ_2. In this case, we proceed as in Case 1 of Theorem 1.3, to conclude that β must be cofinal in the Dehornoy ordering. Thus, we can find an integer k so that $\beta^k > g_i > 1$ for every generator g_i of K_n. Then $g_i \in C$ for all i, and we conclude $C = K_n$.

Case 2. Suppose that β and σ_2 do not commute, and we proceed as in Case 2 of Theorem 1.3. Then, by the subword property of the Dehornoy ordering, we know that $\beta\sigma_2^k\beta^{-1} > 1$ for all $k > 0$, and hence $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k} > 1$ as well. We deduce that $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$ for all $k > 0$ as before. However, the braid $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is not necessarily an element of K_n, but as conjugation by σ_2^6 preserves K_n by Lemma 2.3, we have $\beta\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in K_n$ for all $k > 0$. Hence, the inequality $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$ yields $\beta\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in C$ for all $k > 0$. We conclude that $\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in C$ for all $k > 0$.

Proceeding as in the proof of Theorem 1.3, we may conjugate β by an appropriate (sixth) power of σ_2 to conclude that the convex subgroup C in K_n contains a braid represented by a word of the form $\sigma_2^u\sigma_1 w$, where $u > 1$, and w is a 1-positive, 1-neutral or empty word. Then for each generator g_i of K_n, consider the braid represented by the word $g_i^{-1}\sigma_2^u\sigma_1 w$. As each g_i contains only positive powers of the braids $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$, we see

that $g_i^{-1}\sigma_2^u\sigma_1$ represents a 1-positive braid, by Lemma 2.4. Therefore, the braid $g_i^{-1}\sigma_2^u\sigma_1 w$ is 1-positive, and we conclude that $1 < g_i < \sigma_2^u\sigma_1 w \in C$, hence $g_i \in C$ for all i, and $C = K_n$. $\square$

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